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# Localization forcing and Hechler's theorem for the null ideal (Set Theory and Computability Theory of the Reals)

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# Localization forcing and Hechler's theorem for the null ideal

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## 概要

Hechler の定理とは, 「任意の  $\aleph_1$ -directed な順序集合は, countable chain condition を満たす強制法によって, 順序構造  $(\omega^\omega, \leq^*)$  の中に cofinal に埋め込むことができる」という主張である. 本稿では, 実数直線上のルベグ零集合 (null sets) のなすイデアルにおける集合の包含関係 ( $\subseteq$ ) について, Hechler の定理と同様の「順序構造の埋め込み定理」が成り立つことを示す.

We prove the following theorem: For any  $\aleph_1$ -directed partially ordered set  $Q$ , there is a forcing notion satisfying ccc such that, in the forcing model, there is a basis of the null ideal of the real line which is order-isomorphic to  $Q$  with respect to set-inclusion. This is a variation of Hechler's classical result in the theory of forcing.

## 1 Introduction

For  $f, g \in \omega^\omega$ , we say  $f \leq^* g$  if  $f(n) \leq g(n)$  for all but finitely many  $n < \omega$ . The following theorem, which is due to Hechler [6], is a classical result in the theory of forcing (See also [4]).

**Theorem 1.1.** *Suppose that  $(Q, \leq)$  is a partially ordered set such that every countable subset of  $Q$  has a strict upper bound in  $Q$ , that is, for any countable set  $A \subseteq Q$  there is  $b \in Q$  such that  $a < b$  for all  $a \in A$ . Then there is a forcing notion  $\mathbb{P}$  satisfying ccc such that, in the forcing model by  $\mathbb{P}$ ,  $(\omega^\omega, \leq^*)$  contains a cofinal subset  $\{f_a : a \in Q\}$  which is order-isomorphic to  $Q$ , that is,*

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1. for every  $g \in \omega^\omega$  there is  $a \in Q$  such that  $g \leq^* f_a$ , and
2. for  $a, b \in Q$ ,  $f_a \leq^* f_b$  if and only if  $a \leq b$ .

Soukup [8] asked if the statement of Hechler's theorem holds for the meager ideal or the null ideal of the real line with respect to set-inclusion. Bartoszyński and Kada [3] have answered positively the question for the meager ideal. In the present paper, we will give a positive answer for the null ideal.

## 2 Combinatorial view of null sets

In this section, we review the relationship between Borel null sets of the real line and combinatorics on natural numbers, which is described in [1]. We work in the Cantor space  $2^\omega$  with the standard product measure.

Choose a strictly increasing function  $h \in \omega^\omega$  satisfying  $2^{h(n)-h(n-1)} \geq n+1$  for  $1 \leq n < \omega$  (for example, just let  $h(n) = n^2$ ). For each  $n < \omega$ , let  $\{C_i^n : i < \omega\}$  be a list of all clopen subsets of  $2^\omega$  of measure  $2^{-h(n)}$ . We assume that such  $h$  and  $C_i^n$ 's are fixed throughout this paper.

For a function  $f \in \omega^\omega$ , we define

$$H_f = \bigcap_{N} \bigcup_{n > N} C_{f(n)}^n.$$

Then  $H_f$  is a  $G_\delta$  null set, and every null set  $X$  is covered by  $H_f$  for some  $f \in \omega^\omega$ .

Let  $\mathcal{S} = \prod_{n < \omega} [\omega]^{\leq n}$ . We call each  $\varphi \in \mathcal{S}$  a *slalom*. As in the case of a function, for a slalom  $\varphi \in \mathcal{S}$  we define

$$H_\varphi = \bigcap_{N} \bigcup_{n > N} \bigcup_{i \in \varphi(n)} C_i^n.$$

Then  $H_\varphi$  is a  $G_\delta$  null set, and the following hold:

1. For  $f \in \omega^\omega$  and  $\varphi \in \mathcal{S}$ , if  $f(n) \in \varphi(n)$  holds for all but finitely many  $n < \omega$ , then  $H_f \subseteq H_\varphi$ .
2. For  $\varphi, \psi \in \mathcal{S}$ , if  $\psi(n) \subseteq \varphi(n)$  holds for all but finitely many  $n < \omega$ , then  $H_\psi \subseteq H_\varphi$ .

Note that the reversed implications in the above statements do not hold in general.

Now we define a canonical way to find a nonempty closed set outside  $H_\varphi$ .

For a slalom  $\varphi \in \mathcal{S}$ , define a function  $r_\varphi \in \omega^\omega$  by induction on  $n < \omega$  as follows:  $r_\varphi(0) = 0$ , and for  $1 \leq n < \omega$ , let

$$r_\varphi(n) = \min\{i < \omega : C_i^n \subseteq C_{r_\varphi(n-1)}^{n-1} \setminus \bigcup_{j \in \varphi(n)} C_j^n\}.$$

This induction goes well because, by the choice of  $h$ , we have  $\mu(C_k^{n-1}) \geq (n+1) \cdot \mu(C_j^n)$  for  $j, k < \omega$ .

Let  $R_\varphi = \bigcap_{n < \omega} C_{r_\varphi(n)}^n$ .  $R_\varphi$  is a nonempty closed set, because it is the intersection of a decreasing sequence of closed sets in a compact space. Let  $A_\varphi = \bigcup_{n < \omega} \bigcup_{i \in \varphi(n)} C_i^n$ . Then clearly  $H_\varphi \subseteq A_\varphi$ . By the construction of  $r_\varphi$ , we have  $R_\varphi \cap A_\varphi = \emptyset$ , and hence  $R_\varphi \cap H_\varphi = \emptyset$ .

For  $\varphi, \psi \in \mathcal{S}$ , if  $r_\varphi(n) \in \psi(n)$  for infinitely many  $n < \omega$ , then  $R_\varphi \subseteq H_\psi$  and hence  $H_\psi \not\subseteq H_\varphi$ .

### 3 Localization forcing

In this section, we will introduce a modified form of *localization forcing* LOC, which is defined in [2, Section 3.1].

Let  $\mathcal{T} = \bigcup_{n < \omega} \prod_{i < n} [\omega]^{\leq i}$ . A condition  $p$  of LOC is of the form  $p = (s^p, F^p)$ , where  $s^p \in \mathcal{T}$ ,  $F^p \subseteq \omega^\omega$  and  $|F^p| \leq |s^p|$ . For conditions  $p, q$  in LOC,  $p \leq q$  if  $s^p \supseteq s^q$ ,  $F^p \supseteq F^q$ , and for each  $n \in |s^p| \setminus |s^q|$  and  $f \in F^q$  we have  $f(n) \in s^p(n)$ .

It is easy to see the following.

1. For each  $n < \omega$ , the set  $\{q \in \text{LOC} : |s^q| \geq n\}$  is dense in LOC.
2. For each  $f \in \omega^\omega$ , the set  $\{q \in \text{LOC} : f \in F^q\}$  is dense in LOC.
3. LOC is  $\sigma$ -linked, and hence it satisfies ccc.

Let  $\mathbf{V}$  be a ground model, and  $G$  a LOC-generic filter over  $\mathbf{V}$ . In  $\mathbf{V}[G]$ , let  $\varphi_G = \bigcup\{s^p : p \in G\}$ . Then  $\varphi_G \in \mathcal{S}$  and, for every  $f \in \omega^\omega \cap \mathbf{V}$ , for all but finitely many  $n < \omega$  we have  $f(n) \in \varphi_G(n)$ .

Let  $H_G = H_{\varphi_G}$ . Then in  $\mathbf{V}[G]$ , by the observation in Section 2, for every Borel null set  $X \subseteq 2^\omega$  which is coded in  $\mathbf{V}$ , we have  $X \subseteq H_G$ .

Now we define a modified form of localization forcing.

**Definition 3.1.** Define  $\text{LOC}^*$  as follows. A condition  $p$  of  $\text{LOC}^*$  is of the form  $p = (s^p, w^p, F^p)$ , where

1.  $s^p \in \mathcal{T}$ ,  $w^p < \omega$ ,  $F^p \subseteq \omega^\omega$ , and
2.  $|F^p| \leq w^p \leq |s^p|$ .

For  $p, q \in \text{LOC}^*$ ,  $p \leq q$  if

3.  $s^q \subseteq s^p$ ,  $w^q \leq w^p$ ,  $F^q \subseteq F^p$ , and for  $n \in |s^p| \setminus |s^q|$  and  $f \in F^q$  we have  $f(n) \in s^p(n)$ ;
4.  $w^p \leq w^q + (|s^p| - |s^q|)$ ;
5. For  $n \in |s^p| \setminus |s^q|$ , we have  $|s^p(n)| \leq w^q + (n - |s^q|)$ .

We show that the forcing  $\text{LOC}^*$  has similar properties to  $\text{LOC}$ .

**Lemma 3.2.** For each  $n < \omega$ , the set  $\{q \in \text{LOC}^* : |s^q| \geq n\}$  is dense in  $\text{LOC}^*$ .

*Proof.* Easy. □

**Lemma 3.3.** For each  $f \in \omega^\omega$ , the set  $\{q \in \text{LOC}^* : f \in F^q\}$  is dense in  $\text{LOC}^*$ .

*Proof.* Fix  $p \in \text{LOC}^*$  and  $f \in \omega^\omega$ . Define  $q = (s^q, w^q, F^q)$  as follows:  $|s^q| = |s^p| + 1$ ,  $s^q \restriction |s^p| = s^p$ ,  $s^q(|s^p|) = \{f(|s^p|) : f \in F^p\}$ ,  $w^q = w^p + 1$  and  $F^q = F^p \cup \{f\}$ . It is easy to see that  $q \in \text{LOC}^*$  and  $q \leq p$ . □

**Lemma 3.4.**  $\text{LOC}^*$  is  $\sigma$ -linked, and hence it satisfies ccc.

*Proof.* It is easily seen that the set  $L = \{p \in \text{LOC}^* : w^p \geq 2 \cdot |F^p|\}$  is dense in  $\text{LOC}^*$ . For each  $s \in \mathcal{T}$  and  $w \leq |s|$ , let  $L_{s,w} = \{p \in L : s^p = s \text{ and } w^p = w\}$ . Then  $L = \bigcup \{L_{s,w} : s \in \mathcal{T} \text{ and } w \leq |s|\}$  and, for each  $s \in \mathcal{T}$  and  $w \leq |s|$ , any two conditions in  $L_{s,w}$  are compatible. □

Let  $\mathbf{V}$  be a ground model, and  $G$  a  $\text{LOC}^*$ -generic filter over  $\mathbf{V}$ . In  $\mathbf{V}[G]$ , let  $\varphi_G = \bigcup \{s^p : p \in G\}$ . Then, by Lemmata 3.2 and 3.3, we have  $\varphi_G \in \mathcal{S}$  and, for every  $f \in \omega^\omega \cap \mathbf{V}$ , for all but finitely many  $n < \omega$  we have  $f(n) \in \varphi_G(n)$ .

Let  $H_G = H_{\varphi_G}$ . The following proposition follows from the observation in Section 2.

**Proposition 3.5.** *Let  $\mathbf{V}$  be a ground model and  $G$  a  $\text{LOC}^*$ -generic filter over  $\mathbf{V}$ . Then in  $\mathbf{V}[G]$ , for every Borel null set  $X \subseteq 2^\omega$  which is coded in  $\mathbf{V}$ , we have  $X \subseteq H_G$ .*

As we observed in Section 2, in  $\mathbf{V}[G]$ , we can define  $r_{\varphi_G}$  and  $R_{\varphi_G}$  from  $\varphi_G$ . Note that, in this context, every  $x \in R_{\varphi_G}$  is a random real over  $\mathbf{V}$ . We can naturally define a  $\text{LOC}^*$ -name  $\dot{r}$  for  $r_{\varphi_G}$  so that, for  $p \in \text{LOC}^*$ , if  $|s^p| = n$  then  $p$  decides the value of  $\dot{r} \restriction n$ , because  $r_{\varphi_G} \restriction n$  depends only on  $\varphi_G \restriction n$ .

## 4 Hechler's theorem for the null ideal

In this section, we will construct a ccc forcing notion which yields Hechler's theorem for the null ideal. The idea is to use localization forcing at each step, instead of the dominating real partial order used in Hechler's construction.

Let  $(Q, \leq)$  be a partially ordered set such that every countable subset of  $Q$  has a strict upper bound in  $Q$ , that is, for every countable set  $A \subseteq Q$  there is  $b \in Q$  such that  $a < b$  for all  $a \in A$ . Extend the order to  $Q^* = Q \cup \{Q\}$  by letting  $a < Q$  for all  $a \in Q$ .

Fix a well-founded cofinal subset  $R$  of  $Q$ . Define the rank function on the well-founded set  $R^* = R \cup \{Q\}$  in the usual way. For  $a \in Q \setminus R$ , let  $\text{rank}(a) = \min\{\text{rank}(b) : b \in R^* \text{ and } a < b\}$ . For  $x, y \in Q^*$ , we say  $x \ll y$  if  $x < y$  and  $\text{rank}(x) < \text{rank}(y)$ . For  $x \in Q^*$ , let  $Q_x = \{y \in Q : y \ll x\}$ .

For  $D \subseteq Q$  and  $\xi \leq \text{rank}(Q)$ , let  $D_{<\xi} = \{y \in D : \text{rank}(y) < \xi\}$ ,  $D_\xi = \{y \in D : \text{rank}(y) = \xi\}$ , and for  $x \in Q$  with  $\text{rank}(x) = \xi$ , let  $D_{\leq x} = \{y \in D_\xi : y \leq x\}$ .

For  $D \subseteq Q$ , let  $\bar{D} = \{\text{rank}(x) : x \in D\}$ .

For  $E \subseteq D \subseteq Q$ , we say  $E$  is *downward closed in  $D$*  if, for  $x \in E$  and  $y \in D$  if  $y \leq x$  then  $y \in E$ . When  $E$  is downward closed in  $Q$ , we simply say  $E$  is *downward closed*.

**Definition 4.1.** We define forcing notions  $\mathbb{N}_a$  for  $a \in Q^*$  by induction on  $\text{rank}(a)$ .

A condition  $p$  of  $\mathbb{N}_a$  is of the form  $p = \{(s_x^p, w_x^p, F_x^p) : x \in D^p\}$  with the following:

1.  $D^p$  is a finite subset of  $Q_a$ ;

2. For  $x \in D^p$ ,  $s_x^p \in \mathcal{T}$ ,  $w_x^p < \omega$ ,  $F_x^p$  is a finite set of  $N_x$ -names for functions in  $\omega^\omega$ , and  $|F_x^p| \leq w_x^p$ ;
3. For  $x \in D^p$ ,  $\sum\{w_z^p : z \in D_{\leq x}^p\} \leq |s_x^p|$ ;
4. For  $x, y \in D^p$ , if  $\text{rank}(x) = \text{rank}(y)$  then  $|s_x^p| = |s_y^p|$ .

Throughout this paper, for a condition  $p$  in  $N_a$ , we always use the notation  $D^p$ ,  $s_x^p$ ,  $w_x^p$  and  $F_x^p$  to denote respective components of  $p$ . Also, for  $p \in N_a$  and  $\xi \in \bar{D}^p$ , let  $l_\xi^p$  be the length of  $s_x^p$  for  $x \in D_\xi^p$ .

For  $p \in N_a$  and  $b \in Q_a$ , define  $p \restriction b \in N_b$  by letting  $p \restriction b = \{(s_x^p, w_x^p, F_x^p) : x \in D^p \cap Q_b\}$ .

For conditions  $p, q$  in  $N_a$ ,  $p \leq q$  if:

5.  $D^q \subseteq D^p$ ;
6. For  $x \in D^q$ ,  $s_x^p \supseteq s_x^q$ ,  $w_x^p \geq w_x^q$ ,  $F_x^p \supseteq F_x^q$  and, for all  $n \in |s_x^p| \setminus |s_x^q|$  and  $\dot{f} \in F_x^q$  we have  $p \restriction x \Vdash_{N_x} \dot{f}(n) \in s_x^p(n)$ ;
7. For  $\xi \in \bar{D}^q$  and  $x, y \in D_\xi^q$ , if  $x < y$ , then for all  $n \in l_\xi^p \setminus l_\xi^q$  we have  $s_x^p(n) \subseteq s_y^p(n)$ ;
8. For  $\xi \in \bar{D}^q$ ,  $\sum\{w_x^p : x \in D_\xi^p\} \leq \sum\{w_x^q : x \in D_\xi^q\} + (l_\xi^p - l_\xi^q)$ ;
9. For  $\xi \in \bar{D}^q$ ,  $E \subseteq D_\xi^q$  which is downward closed in  $D_\xi^q$  and  $n \in l_\xi^p \setminus l_\xi^q$ , we have

$$|\bigcup\{s_x^p(n) : x \in E\}| \leq \sum\{w_x^q : x \in E\} + (n - l_\xi^q).$$

*Remark 1.* If  $p \leq q$ , then for any  $\xi \in \bar{D}^q$  and  $E \subseteq D_\xi^p$  we can discard the terms with indices not in  $E$  from both sides of the inequality in clause 8 (using  $w_x^p \geq w_x^q$  from clause 6) to get

$$\sum\{w_x^p : x \in E\} \leq \sum\{w_x^q : x \in E \cap D_\xi^q\} + (l_\xi^p - l_\xi^q).$$

We now verify that Definition 4.1 does indeed define a partial order. (Reflexivity is clear, but we need to prove transitivity.) The simple observation in part (c) of the following proposition justifies not mentioning  $a$  in the notation  $\leq$  for the order relation on  $N_a$ .

**Proposition 4.2.** *We have the following properties.*

- (a) For any conditions  $p, q \in N_a$ , if  $p \leq q$  then for any  $b \in Q_a$ ,  $p \restriction b \leq q \restriction b$ .

(b) The order relation on  $N_a$  is transitive.

(c) For any  $a, b \in Q^*$ , if  $p, q \in N_a \cap N_b$ , then  $p \leq q$  in  $N_a$  if and only if  $p \leq q$  in  $N_b$ .

*Proof.* (a) and (b) are proven simultaneously by induction on the rank of  $a$ . Note that part (b) of the induction hypothesis ensures that for  $p, q \in N_a$  and  $x \in D^q \subseteq Q_a$ ,  $N_x$  is a well-defined partial order and hence the last part of clause 6 makes sense.

(a) All but the last part of clause 6 and clause 8 in the definition of  $p \upharpoonright b \leq q \upharpoonright b$  are inherited directly from the corresponding clauses for  $p \leq q$ . The last part of clause 6 holds because for  $x \in D^{qb} = D^q \cap Q_b$ ,  $(p \upharpoonright b) \upharpoonright x = p \upharpoonright x$ . There remains to check clause 8. Let  $\xi \in \bar{D}^{qb}$ . Using clause 8 for  $p \leq q$  and the fact that  $w_x^p \geq w_x^q$  whenever both are defined, we have

$$\begin{aligned} \sum \{w_x^{pb} : x \in D_\xi^{pb}\} &= \sum \{w_x^p : x \in D_\xi^{pb}\} \\ &= \sum \{w_x^p : x \in D_\xi^p\} - \sum \{w_x^p : x \in D_\xi^p \setminus Q_b\} \\ &\leq \sum \{w_x^q : x \in D_\xi^q\} + (l_\xi^p - l_\xi^q) - \sum \{w_x^p : x \in D_\xi^p \setminus Q_b\} \\ &\leq \sum \{w_x^q : x \in D_\xi^q\} + (l_\xi^p - l_\xi^q) - \sum \{w_x^q : x \in D_\xi^q \setminus Q_b\} \\ &= \sum \{w_x^{qb} : x \in D_\xi^{qb}\} + (l_\xi^{pb} - l_\xi^{qb}). \end{aligned}$$

(b) Suppose that  $a \in Q^*$ ,  $p, q, r \in N_a$  and  $p \leq q \leq r$ . We must show  $p \leq r$ .

For the last part of clause 6, suppose we have  $x \in D_\gamma^r$ ,  $n \in l_\gamma^p \setminus l_\gamma^r$ ,  $\dot{f} \in F_x^r$ . If  $n \in l_\gamma^p \setminus l_\gamma^q$ , then because  $\dot{f} \in F_x^r \subseteq F_x^q$ , the fact that  $p \leq q$  gives  $p \upharpoonright x \Vdash_{N_x} \dot{f}(n) \in s_x^p(n)$ . If  $n \in l_\gamma^q \setminus l_\gamma^r$ , then the fact that  $q \leq r$  gives  $q \upharpoonright x \Vdash_{N_x} \dot{f}(n) \in s_x^q(n)$ . We have  $s_x^p(n) = s_x^q(n)$  by the first part of clause 6 for  $p \leq q$ . Also,  $p \upharpoonright x \leq q \upharpoonright x$  by part (a). Thus,  $p \upharpoonright x \Vdash_{N_x} \dot{f}(n) \in s_x^p(n)$ . We now check clause 9 and leave the other clauses for the reader. Fix  $\xi \in \bar{D}^r$ ,  $E \subseteq D_\xi^r$  which is downward closed in  $D_\xi^r$  and  $n \in l_\xi^p \setminus l_\xi^r$ . Let  $E^q$  be the downward closure of  $E$  in  $D_\xi^q$ . If  $n \in l_\xi^q \setminus l_\xi^r$ , then

$$\begin{aligned} |\bigcup \{s_x^p(n) : x \in E\}| &= |\bigcup \{s_x^q(n) : x \in E\}| \\ &\leq \sum \{w_x^r : x \in E\} + (n - l_\xi^r) \end{aligned}$$

because of clause 9 for  $q \leq r$ . If  $n \in l_\xi^p \setminus l_\xi^q$ , then

$$\begin{aligned} |\bigcup \{s_x^p(n) : x \in E\}| &\leq |\bigcup \{s_x^p(n) : x \in E^q\}| \\ &\leq \sum \{w_x^q : x \in E^q\} + (n - l_\xi^q) \\ &\leq \sum \{w_x^r : x \in E\} + (l_\xi^q - l_\xi^r) + (n - l_\xi^q) \\ &= \sum \{w_x^r : x \in E\} + (n - l_\xi^r). \end{aligned}$$



The second inequality follows from clause 9 for  $p \leq q$  and the third from Remark 1 for  $q \leq r$ . Hence we have  $p \leq r$ .

(c) The definition of the order on  $N_a$  makes no mention of  $a$ .  $\square$

**Definition 4.3.** For a downward closed set  $A \subseteq Q$ , let  $N_A = \{p \in N_Q : D^p \subseteq A\}$ , and for  $p \in N_Q$ , we define  $p \restriction A \in N_A$  by letting  $p \restriction A = \{(s_x^p, w_x^p, F_x^p) : x \in D^p \cap A\}$ . For  $\xi \leq \text{rank}(Q)$ , let  $N_\xi = N_{Q_{<\xi}}$  and  $p \restriction \xi = p \restriction Q_{<\xi}$ . Also, for  $\xi \leq \text{rank}(Q)$ , let  $p \restriction \{\xi\} = \{(s_x^p, w_x^p, F_x^p) : x \in D^p\} \in N_{\xi+1}$  and  $p \restriction [\xi, \infty) = \{(s_x^p, w_x^p, F_x^p) : x \in D^p \setminus Q_{<\xi}\} \in N_Q$ .

In this notation,  $N_a = N_{Q_a}$  for  $a \in Q$ , and  $N_Q$  has the same meaning if we consider the subscript  $Q$  either as an element of  $Q^*$  or as a subset of  $Q$ .

Clearly  $A \subseteq B \subseteq Q$  implies  $N_A \subseteq N_B \subseteq N_Q$ . We are going to prove that, if  $A \subseteq B$ , then  $N_A$  is completely embedded into  $N_B$ . This is a fundamental principle of the iterated forcing.

The following lemma, which is a special case of this principle, is easily checked.

**Lemma 4.4.** *If  $B$  is a downward closed subset of  $Q$ ,  $\xi \leq \text{rank}(Q)$ ,  $p \in N_B$  and  $q \in N_{B_{<\xi}}$  extends  $p \restriction \xi$ , then  $q \cup p \restriction [\xi, \infty)$  belongs to  $N_B$  and extends both  $p$  and  $q$ . In particular,  $N_{B_{<\xi}}$  is completely embedded into  $N_B$ .*

Using this lemma, we prove the following.

**Lemma 4.5.** *For downward closed sets  $A, B \subseteq Q$ , if  $A \subseteq B$ , then  $N_A$  is completely embedded into  $N_B$  by the identity map.*

*Proof.* It is easy to see that the compatibility of conditions in  $N_A$  is the same either in  $N_A$  or in  $N_B$ . We show that, for  $p \in N_B$  and  $r \in N_A$ , if  $r \leq p \restriction A$  then there is  $q \in N_B$  satisfying  $q \leq p$  and  $q \leq r$ . We will proceed by induction on  $\sup \bar{A}$ .

Suppose that  $p \in N_B$ ,  $r \in N_A$  and  $r \leq p \restriction A$ . Let  $\gamma = \max \bar{D}^r$ . By the induction hypothesis, there is  $q_{<\gamma} \in N_{B_{<\gamma}}$  satisfying  $q_{<\gamma} \leq p \restriction \gamma$  and  $q_{<\gamma} \leq r \restriction \gamma$ .

For  $x \in D_\gamma^r$ , let  $(s_x, w_x, F_x) = (s_x^r, w_x^r, F_x^r)$ . For  $x \in D_\gamma^p \setminus D_\gamma^r$ , let  $(s_x, w_x, F_x) = (s_x^p, w_x^p, F_x^p)$ .

Let

$$L = \sum \{w_x : x \in D_\gamma^p \cup D_\gamma^r\} + l_\gamma^r.$$

By the induction hypothesis, for each  $x \in D_\gamma^p \cup D_\gamma^r$ ,  $N_x$  is completely embedded into  $N_{B_{<\gamma}}$  and so each  $f \in F_x$  is an  $N_{B_{<\gamma}}$ -name. Choose

$q^* \in \mathbb{N}_{B < \gamma}$  so that  $q^* \leq q_{< \gamma}$  and  $q^*$  decides the values of  $\dot{f} \upharpoonright L$  for all  $\dot{f} \in \bigcup \{F_x : x \in D_\gamma^p \cup D_\gamma^r\}$ . For  $x \in D_\gamma^p \cup D_\gamma^r$  and  $n \in L \setminus |s_x|$ , let  $K_{x,n} \subseteq \omega$  be the set satisfying  $q^* \Vdash K_{x,n} = \{\dot{f}(n) : \dot{f} \in F_x\}$ .

Define  $s_x^*$  for  $x \in D_\gamma^p \cup D_\gamma^r$  in the following way: If  $x \in D_\gamma^r$ , then  $|s_x^*| = L$ ,  $s_x^* \upharpoonright l_\gamma^r = s_x$ , and for  $n \in L \setminus l_\gamma^r$ ,

$$s_x^*(n) = \bigcup \{K_{z,n} : z \in D_{\leq x}^r\}.$$

If  $x \in D_\gamma^p \setminus D_\gamma^r$ , then  $|s_x^*| = L$ ,  $s_x^* \upharpoonright l_\gamma^p = s_x$ , and for  $n \in L \setminus l_\gamma^p$ ,

$$s_x^*(n) = \begin{cases} \bigcup \{s_z(n) : z \in D_{\leq x}^p \cap D_\gamma^r\} \cup \bigcup \{K_{z,n} : z \in D_{\leq x}^p \setminus D_\gamma^r\} & \text{if } l_\gamma^p \leq n < l_\gamma^r \\ \bigcup \{K_{z,n} : z \in (D_\gamma^p \cup D_\gamma^r)_{\leq x}\} & \text{if } l_\gamma^r \leq n < L, \gamma \in \bar{D}^{AA} \\ \bigcup \{K_{z,n} : z \in D_{\leq x}^p\} & \text{if } l_\gamma^r \leq n < L, \gamma \notin \bar{D}^{AA} \end{cases}$$

Now we define  $q = \{(s_x^q, w_x^q, F_x^q) : x \in D^q\}$  by the following:

1.  $D^q = D^p \cup D^{q^*} \cup D_\gamma^r$ ;
2. For  $x \in D^{q^*}$ ,  $(s_x^q, w_x^q, F_x^q) = (s_x^{q^*}, w_x^{q^*}, F_x^{q^*})$ ;
3. For  $x \in D_\gamma^p \cup D_\gamma^r$ ,  $(s_x^q, w_x^q, F_x^q) = (s_x^*, w_x^*, F_x^*)$ ;
4. For  $x \in D^p \setminus Q_{< \gamma+1}$ ,  $(s_x^q, w_x^q, F_x^q) = (s_x^p, w_x^p, F_x^p)$ .

We now check that  $q \in \mathbb{N}_B$ . The conditions of Definition 4.1 are satisfied below (resp. above) rank  $\gamma$  because  $q^*$  (resp.  $p$ ) is a condition. Consider what they say at rank  $\gamma$ . The first clause is trivial. The fourth holds because the  $s_x^q$ 's all have domain  $L$ . The third clause can be checked in two cases.

- (i) If  $x \in D_\gamma^r$ , then  $D_{\leq x}^q = (D^p \cup D^r)_{\leq x} = D_{\leq x}^r$ , so  $\sum \{w_z^q : z \in D_{\leq x}^q\} = \sum \{w_z^r : z \in D_{\leq x}^r\} \leq l_\gamma^r \leq L$ .
- (ii) If  $x \in D_\gamma^p \setminus D_\gamma^r$ , then  $D_{\leq x}^q = D_{\leq x}^p \cup D_{\leq x}^r$ , so  $\sum \{w_z^q : z \in D_{\leq x}^q\} = \sum \{w_z : z \in D_{\leq x}^p \cup D_{\leq x}^r\} \leq L$ .

For the second, all the requirements except that the  $s_x^q$ 's are partial slaloms follow from the fact that  $p$  and  $r$  are conditions. We need to check that  $|s_x^*(n)| \leq n$  for each relevant  $n$ . If  $x \in D_\gamma^r$ , then for  $l_\gamma^r \leq n < L$ , we have  $|s_x^*(n)| \leq \sum \{w_z^r : z \in D_{\leq x}^r\} \leq |s_x^r| = l_\gamma^r \leq n$ . If  $x \in D_\gamma^p \setminus D_\gamma^r$ , we consider four cases.

Case 1.  $l_\gamma^p \leq n < l_\gamma^r$  and  $\gamma \in \bar{D}^{pA}$ . Definition 4.1(9) for  $r \leq p \upharpoonright A$  with  $E = D_{\leq x}^p \cap D_\gamma^r$  gives

$$\begin{aligned} |s_x^*(n)| &\leq \sum \{w_z^p : z \in E\} + (n - l_\gamma^p) + \sum \{w_z^p : z \in D_{\leq x}^p \setminus E\} \\ &= \sum \{w_z^p : z \in D_{\leq x}^p\} + (n - l_\gamma^p) \\ &\leq l_\gamma^p + (n - l_\gamma^p) = n. \end{aligned}$$

Case 2.  $l_\gamma^p \leq n < l_\gamma^r$  and  $\gamma \notin \bar{D}^{pA}$ . In this case,  $D_{\leq x}^p \cap D_\gamma^r \subseteq D_\gamma^p \cap A = \emptyset$ , so  $|s_x^*(n)| \leq \sum \{w_z^p : z \in D_{\leq x}^p\} \leq l_\gamma^p \leq n$ .

Case 3.  $l_\gamma^r \leq n < L$  and  $\gamma \in \bar{D}^{pA}$ . Definition 4.1(8) for  $r \leq p \upharpoonright A$  gives  $\sum \{w_z^r : z \in D_\gamma^r\} \leq \sum \{w_z^p : z \in D_\gamma^{pA}\} + (l_\gamma^r - l_\gamma^p)$ . Removing terms with  $z \not\leq x$  from both sides (see Remark 1) gives

$$\sum \{w_z^r : z \in D_{\leq x}^r\} \leq \sum \{w_z^p : z \in D_{\leq x}^p \cap A\} + (l_\gamma^r - l_\gamma^p).$$

From the formula for  $s_x^*(n)$  we now get

$$\begin{aligned} |s_x^*(n)| &\leq \sum \{w_z^r : z \in D_{\leq x}^r\} + \sum \{w_z^p : z \in D_{\leq x}^p \setminus A\} \\ &\leq \sum \{w_z^p : z \in D_{\leq x}^p \cap A\} + (l_\gamma^r - l_\gamma^p) + \sum \{w_z^p : z \in D_{\leq x}^p \setminus A\} \\ &= \sum \{w_z^p : z \in D_{\leq x}^p\} + (l_\gamma^r - l_\gamma^p) \\ &\leq l_\gamma^p + (l_\gamma^r - l_\gamma^p) = l_\gamma^r \leq n. \end{aligned}$$

Case 4.  $l_\gamma^r \leq n < L$  and  $\gamma \notin \bar{D}^{pA}$ . In this case we have  $|s_x^*(n)| \leq \sum \{w_z^p : z \in D_{\leq x}^p\} \leq l_\gamma^p \leq n$ .

Thus,  $q$  is a condition.

We now check Definition 4.1(5–9) for  $q \leq r$  and  $q \leq p$ . Clause 5 follows from the definition of  $q$ . For clauses 6–9, first note that below rank  $\gamma$ , they hold because  $q^* \leq p \upharpoonright \gamma$  and  $q^* \leq r \upharpoonright \gamma$ . Consider what happens at rank  $\gamma$ . Clause 6 holds because for  $x \in D_\gamma^p \cup D_\gamma^r$  and all the relevant values of  $\dot{f}$  and  $n$ , we have from the definitions that  $q^* \Vdash \dot{f}(n) \in K_{x,n}$  and  $K_{x,n} \subseteq s_x^*(n)$ . For clause 7, we consider three cases. Let  $x < y$  be elements of  $D_\gamma^p \cup D_\gamma^r$ .

- (i) If  $x, y \in D_\gamma^r$ , then for checking  $q \leq r$ , just use the monotonicity of  $s_x^*(n)$  as a function of  $x$ . For checking  $q \leq p$  (so now we assume  $x, y \in D_\gamma^p$  as well), we also need to consider values of  $n$  such that  $l_\gamma^p \leq n < l_\gamma^r$ . But then  $s_x^*(n) = s_x^r(n) \subseteq s_y^r(n) = s_y^*(n)$  because  $r \leq p \upharpoonright A$ .

This is the only case to consider for checking clause 7 for  $q \leq r$  at stage  $\gamma$ . The remaining cases deal with checking  $q \leq p$ . Note that if  $y \in D_\gamma^r \cap D_\gamma^p = D_\gamma^p \cap A$  then also  $x \in D_\gamma^r \cap D_\gamma^p$  since  $A$  is downward closed.

- (ii) If  $x, y \in D_\gamma^p \setminus D_\gamma^r$ , use the monotonicity of  $s_x^*(n)$  as a function of  $x$ .
- (iii) If  $x \in D_\gamma^r \cap D_\gamma^p$  and  $y \in D_\gamma^p \setminus D_\gamma^r$ , then consider first a value of  $n$  such that  $l_\gamma^p \leq n < l_\gamma^r$ . We have  $s_x^*(n) = s_x(n) \subseteq \bigcup \{s_z(n) : z \in D_{\leq y}^p \cap D_\gamma^r\} \subseteq s_y^*(n)$ . Next consider  $n$  such that  $l_\gamma^r \leq n < L$ . We have  $s_x^*(n) = \bigcup \{K_{z,n} : z \in D_{\leq x}^r\} \subseteq \bigcup \{K_{z,n} : z \in (D_\gamma^p \cup D_\gamma^r)_{\leq y}\} = s_y^*(n)$ .

This takes care of clause 7. Clause 8 follows from the fact that from the definition of  $L$  we have  $\sum \{w_x : x \in D_\gamma^r \cup D_\gamma^p\} \leq L - l_\gamma^r \leq L - l_\gamma^p$ . For clause 9, first we check  $q \leq r$ . If  $E \subseteq D_\gamma^r$  is downward closed in  $D_\gamma^r$  and  $l_\gamma^r \leq n < L$ , then  $|\bigcup \{s_x^*(n) : x \in E\}| = |\bigcup \{K_{x,n} : x \in E\}| \leq \sum \{w_x^r : x \in E\}$ . Next we check  $q \leq p$ . Suppose  $\gamma \in \bar{D}^{pA}$  and let  $E \subseteq D_\gamma^p$  be downward closed. Consider four cases.

Case 1.  $l_\gamma^p \leq n < l_\gamma^r$  and  $\gamma \in \bar{D}^{pA}$ . Using Definition 4.1(9) for  $r \leq p \upharpoonright A$  and the fact that  $E \cap A$  is downward closed in  $D^{pA}$ , we have

$$\begin{aligned} |\bigcup \{s_x^*(n) : x \in E\}| &= |\bigcup \{s_x^*(n) : x \in E \cap A\} \cup \bigcup \{s_x^*(n) : x \in E \setminus A\}| \\ &= |\bigcup \{s_x^r(n) : x \in E \cap A\} \cup \bigcup \{K_{x,n} : x \in E \setminus A\}| \\ &\leq \sum \{w_x^{pA} : x \in E \cap A\} + (n - l_\gamma^p) + \sum \{w_x^p : x \in E \setminus A\} \\ &= \sum \{w_x^p : x \in E\} + (n - l_\gamma^p). \end{aligned}$$

Case 2.  $l_\gamma^p \leq n < l_\gamma^r$  and  $\gamma \notin \bar{D}^{pA}$ . Then  $E \cap A = \emptyset$ , and the calculation for case 1 reduces to

$$\begin{aligned} |\bigcup \{s_x^*(n) : x \in E\}| &= |\bigcup \{s_x^*(n) : x \in E \setminus A\}| \\ &= |\bigcup \{K_{x,n} : x \in E \setminus A\}| \\ &\leq \sum \{w_x^p : x \in E \setminus A\} \\ &\leq \sum \{w_x^p : x \in E\} + (n - l_\gamma^p). \end{aligned}$$

Case 3.  $l_\gamma^r \leq n < L$  and  $\gamma \in \bar{D}^{pA}$ . Let  $E^r$  be the downward closure in  $D_\gamma^r$  of  $E \cap A = E \cap D_\gamma^{pA}$ . Using Definition 4.1(8) for  $r \leq p \upharpoonright A$  and removing terms with  $z \notin E^r$  from both sides gives

$$\sum \{w_z^r : z \in E^r\} \leq \sum \{w_z^p : z \in E \cap A\} + (l_\gamma^r - l_\gamma^p).$$

Then we get

$$\begin{aligned} |\bigcup \{s_x^*(n) : x \in E\}| &= |\bigcup \{K_{z,n} : z \in E^r\} \cup \bigcup \{K_{z,n} : z \in E \setminus D_\gamma^r\}| \\ &\leq \sum \{w_z^r : z \in E^r\} + \sum \{w_z^p : z \in E \setminus A\} \\ &\leq \sum \{w_z^p : z \in E \cap A\} + (n - l_\gamma^p) + \sum \{w_z^p : z \in E \setminus A\} \\ &\leq \sum \{w_z^p : z \in E\} + (n - l_\gamma^p). \end{aligned}$$

Case 4.  $l_\gamma^r \leq n < L$  and  $\gamma \notin \bar{D}^{nA}$ . We have

$$|\bigcup\{s_x^*(n) : x \in E\}| = |\bigcup\{K_{z,n} : z \in E\}| \leq \sum\{w_z^p : z \in E\}.$$

Thus,  $q \leq r$ . The proof that  $q \leq p$  is completed by appealing to Lemma 4.4.  $\square$

The following definition and lemma provide a simple mechanism for extending conditions.

**Definition 4.6.** Let  $B \subseteq Q$  be a downward closed set and  $\gamma \in \bar{B}$ .  $p' = \{(s_x^{p'}, w_x^{p'}, F_x^{p'}) : x \in D^{p'}\}$  is a  $\gamma$ -precondition of  $\mathbb{N}_B$  if  $p'$  satisfies the following:

1.  $D^{p'}$  is a finite subset of  $B$ ;
2. For  $x \in D^{p'}$ ,  $s_x^{p'} \in \mathcal{T}$ ,  $w_x^{p'} < \omega$ ,  $F_x^{p'}$  is a finite set of  $\mathbb{N}_x$ -names for functions in  $\omega^\omega$ , and  $|F_x^{p'}| \leq w_x^{p'}$ ;
- 3'. For  $x \in D^{p'} \setminus D_\gamma^{p'}$ ,  $\sum\{w_z^{p'} : z \in D_{\leq x}^{p'}\} \leq |s_x^{p'}|$ ;
4. For  $x, y \in D^{p'}$ , if  $\text{rank}(x) = \text{rank}(y)$  then  $|s_x^{p'}| = |s_y^{p'}|$ .

For  $\xi \in \bar{D}^{p'}$ , we will let  $l_\xi^{p'}$  be the length of  $s_x^{p'}$  for  $x \in D_\xi^{p'}$ .

For  $\gamma$ -precondition  $p'$  of  $\mathbb{N}_B$  and  $p \in \mathbb{N}_B$ , we say  $p'$  is a  $\gamma$ -preextension of  $p$  if

1.  $D^{p'} \supseteq D^p$  and  $D^{p'} \setminus Q_{<\gamma+1} = D^p \setminus Q_{<\gamma+1}$ ;
2.  $p' \restriction \gamma \leq p \restriction \gamma$ ;
3. For  $x \in D_\gamma^p$ ,  $s_x^{p'} = s_x^p$ ,  $F_x^{p'} = F_x^p$  and  $w_x^{p'} \geq w_x^p$ ;
4. For  $x \in D_\gamma^{p'} \setminus D_\gamma^p$ ,  $F_x^{p'} = \emptyset$  and  $w_x^{p'} = 0$ ;
5. For  $x \in D^p \setminus Q_{<\gamma+1}$ ,  $(s_x^{p'}, w_x^{p'}, F_x^{p'}) = (s_x^p, w_x^p, F_x^p)$ .

**Lemma 4.7.** Let  $B \subseteq Q$  be a downward closed set,  $p \in \mathbb{N}_B$ ,  $\gamma \in \bar{B}$ ,  $p' = \{(s_x^{p'}, w_x^{p'}, F_x^{p'}) : x \in D^{p'}\}$  a  $\gamma$ -preextension of  $p$  such that  $D_\gamma^{p'} \neq \emptyset$ , and  $N < \omega$ . Then there is  $q \in \mathbb{N}_B$  such that:

1.  $q \leq p$  and  $q \restriction \gamma \leq p' \restriction \gamma$ ;
2.  $D_\gamma^q = D_\gamma^{p'}$  and, for  $x \in D_\gamma^q$ ,  $s_x^q \supseteq s_x^{p'}$ ,  $w_x^q = w_x^{p'}$  and,  $F_x^q = F_x^{p'}$ ;

3.  $D^q \setminus Q_{<\gamma+1} = D^p \setminus Q_{<\gamma+1}$  and, for  $x \in D^q \setminus Q_{<\gamma+1}$ ,  $s_x^q = s_x^p$ ,  $w_x^q = w_x^p$  and  $F_x^q = F_x^p$ ;

4.  $l_\gamma^q \geq N$ .

*Proof.* Let  $L = \max\{\sum\{w_x^{p'} : x \in D_\gamma^{p'}\} + l_\gamma^{p'}, N\}$ .

Note that clause 3 in the definition of “ $p'$  is a  $\gamma$ -preextension of  $p$ ” ensures that  $l_\gamma^{p'} = l_\gamma^p$  as long as the latter is defined, i.e., as long as  $\gamma \in \bar{D}^p$ .

Using Lemma 4.5, choose  $q^* \in \mathbb{N}_{B<\gamma}$  so that  $q^* \leq p' \restriction \gamma$  and  $q^*$  decides the values of  $\dot{f} \restriction L$  for all  $\dot{f} \in \bigcup\{F_x^{p'} : x \in D_\gamma^{p'}\} = \bigcup\{F_x^p : x \in D_\gamma^p\}$ . For  $x \in D_\gamma^p$  and  $n \in L \setminus l_\gamma^{p'} = L \setminus l_\gamma^p$ , let  $K_{x,n} \subseteq \omega$  be the set satisfying  $q^* \Vdash K_{x,n} = \{\dot{f}(n) : \dot{f} \in F_x^p\}$ . Note that  $|K_{x,n}| \leq |F_x^p| \leq w_x^p$ .

Define  $s_x$  for  $x \in D_\gamma^{p'}$  as follows:  $|s_x| = L$ ,  $s_x \restriction l_\gamma^{p'} = s_x^{p'}$ , and for  $n \in L \setminus l_\gamma^{p'}$ , if  $x \in D_\gamma^p$  then  $s_x(n) = \bigcup\{K_{z,n} : z \in D_{\leq x}^p\}$  and if  $x \notin D_\gamma^p$  then  $s_x(n) = \emptyset$ . Now we define  $q = \{(s_x^q, w_x^q, F_x^q) : x \in D^q\}$  as follows:

1.  $D^q = D^{q^*} \cup D^{p'}$ ;
2. For  $x \in D^{q^*}$ ,  $(s_x^q, w_x^q, F_x^q) = (s_x^{q^*}, w_x^{q^*}, F_x^{q^*})$ ;
3. For  $x \in D_\gamma^{p'}$ ,  $(s_x^q, w_x^q, F_x^q) = (s_x, w_x^{p'}, F_x^{p'})$ ;
4. For  $x \in D^q \setminus Q_{<\gamma+1}$ ,  $(s_x^q, w_x^q, F_x^q) = (s_x^{p'}, w_x^{p'}, F_x^{p'})$ .

We now need to check that  $q \in \mathbb{N}_B$  and  $q$  satisfies the requirement. For  $x \in D_\gamma^{p'}$ ,  $l_\gamma^{p'} \leq n < L$ , we check that  $|s_x(n)| \leq n$  and leave the rest of the verification to the reader. If  $x \notin D_\gamma^p$ , then  $s_x(n) = \emptyset$ . Suppose now that  $x \in D_\gamma^p$ . Then  $|s_x(n)| = |\bigcup\{K_{z,n} : z \in D_{\leq x}^p\}| \leq \sum\{w_z^p : z \in D_{\leq x}^p\} \leq |s_x^p| = l_\gamma^p = l_\gamma^{p'} \leq n$ .  $\square$

Next we prove that  $\mathbb{N}_Q$  satisfies ccc.

**Lemma 4.8.** *Let  $W$  be the collection of conditions  $q \in \mathbb{N}_Q$  satisfying the following properties:*

1. For all  $x \in D^q$ ,  $2 \cdot |F_x^q| \leq w_x^q$ ;
2. For all  $\xi \in \bar{D}^q$ ,  $2 \cdot \sum\{w_x^q : x \in D_\xi^q\} \leq l_\xi^q$ .

*Then  $W$  is dense in  $\mathbb{N}_Q$ .*

*Proof.* By induction on  $\xi \leq \text{rank}(Q)$ , we will show that  $W_{<\xi}$  is dense in  $N_\xi$ .

Fix  $p \in N_\xi$  and let  $\gamma = \max \bar{D}^p$ . Define a  $\gamma$ -preextension  $p'$  of  $p$  by the following:  $D^{p'} = D^p$ ,  $p' \upharpoonright \gamma = p \upharpoonright \gamma$  and, for  $x \in D_\gamma^p$ ,  $s_x^{p'} = s_x^p$ ,  $F_x^{p'} = F_x^p$  and  $w_x^{p'} = \max\{w_x^p, 2 \cdot |F_x^p|\}$ . Let  $N = 2 \cdot \sum\{w_x^{p'} : x \in D_\gamma^p\}$ . Applying Lemma 4.7 to  $p$ ,  $p'$  and  $N$ , we get a condition  $q \leq p$  as in the lemma. By induction hypothesis, there is a condition  $q^* \in W_{<\gamma}$ ,  $q^* \leq q \upharpoonright \gamma$ . Then  $q^* \cup q \upharpoonright \{\gamma\}$  extends  $q$  (by Lemma 4.4) and belongs to  $W_{<\gamma+1}$ .  $\square$

**Lemma 4.9.**  $N_Q$  satisfies ccc.

*Proof.* Let  $W$  be the dense set given by Lemma 4.8. If  $A \subseteq W$  is uncountable, then thin  $A$  out to an uncountable set  $A' \subseteq A$  such that

- (1)  $\{\bar{D}^p : p \in A'\}$  is a  $\Delta$ -system with root  $u$ ;
- (2) For  $\xi \in u$ , there is an  $l_\xi$  such that  $l_\xi^p = l_\xi$  for all  $p \in A'$ ;
- (3)  $\{D^p : p \in A'\}$  is a  $\Delta$ -system with root  $U$ ;
- (4) For  $x \in U$ , there are  $s_x$  and  $w_x$  such that  $s_x^p = s_x$  and  $w_x^p = w_x$  for all  $p \in A'$ ;
- (5) For each  $U' \subseteq U$ , there is a number  $k_{U'}$  such that for each  $p \in A'$ ,

$$\sum\{|F_z^p| : \text{for some } x \in U', z \in D_{\leq x}^p\} = k_{U'}.$$

Note that, because  $p \in W$ , we have  $2k_{U'} \leq \sum\{w_z^p : \text{for some } x \in U', z \in D_{\leq x}^p\}$ .

Let  $p$  and  $q$  be any two conditions in  $A'$ . Let  $\xi_0 < \xi_1 < \dots < \xi_{k-1}$  be the increasing enumeration of  $\bar{D}^p \cup \bar{D}^q$ . We will inductively define conditions  $r_i \in N_{<\xi_i+1}$ ,  $i < k$ , so that

1.  $r_i$  is a common extension of  $p \upharpoonright (\xi_i + 1)$  and  $q \upharpoonright (\xi_i + 1)$ ;
2. For each  $i < k - 1$ ,  $r_{i+1} \upharpoonright \xi_{i+1} \leq r_i$ .

Set  $r_{-1} = \emptyset$ . When  $\xi_i \notin u$ , then only one of  $\bar{D}^p$ ,  $\bar{D}^q$  contains  $\xi_i$ . If  $\xi_i \in \bar{D}^p \setminus \bar{D}^q$ , then let  $r_i = r_{i-1} \cup p \upharpoonright \{\xi_i\}$ . Then  $r_i$  inherits from  $r_{i-1}$  and  $p \upharpoonright \{\xi_i\}$  the properties needed for being a condition. It extends  $p \upharpoonright (\xi_i + 1)$  by Lemma 4.4. It extends  $q \upharpoonright (\xi_i + 1)$  because the inclusion of the domains holds and  $q \upharpoonright (\xi_i + 1) = q \upharpoonright (\xi_{i-1} + 1)$ , so the relevant values of  $x$  and  $\xi$  for which  $x \in D^q$  or  $\xi \in \bar{D}^q$  in clauses 6–9 of Definition 4.1 applied to  $r_i \leq q \upharpoonright (\xi_i + 1)$  all have rank at most  $\xi_{i-1}$  and hence the clauses hold because  $r_{i-1} \leq q \upharpoonright (\xi_{i-1} + 1)$ . Similarly if  $\xi_i \in \bar{D}^q \setminus \bar{D}^p$ .

Now suppose  $\xi_i = \gamma \in u$ . Proceed as follows.

- (a) Let  $L = \sum\{w_x^p : x \in D_\gamma^p\} + \sum\{w_x^q : x \in D_\gamma^q\} + l_\gamma$ .
- (b) Get  $r^* \in \mathbb{N}_{<\gamma_i}$ ,  $r^* \leq r_{i-1}$  which decides the values of  $\dot{f} \upharpoonright L$  for  $f \in F_x^p$ ,  $x \in D_\gamma^p$  and  $f \in F_x^q$ ,  $x \in D_\gamma^q$ . For  $n \in L$ , let  $K_{x,n}$  be the set such that
- (i)  $r^* \Vdash \{\dot{f}(n) : \dot{f} \in F_x^p\} = K_{x,n}$ , if  $x \in D_\gamma^p \setminus D_\gamma^q$ ;
  - (ii)  $r^* \Vdash \{\dot{f}(n) : \dot{f} \in F_x^q\} = K_{x,n}$ , if  $x \in D_\gamma^q \setminus D_\gamma^p$ ;
  - (iii)  $r^* \Vdash \{\dot{f}(n) : \dot{f} \in F_x^p \cup F_x^q\} = K_{x,n}$ , if  $x \in D_\gamma^p \cap D_\gamma^q$ .

Note that

$$\begin{aligned} |\bigcup\{K_{x,n} : x \in D_\gamma^p \cup D_\gamma^q\}| &\leq \sum\{w_x^p : x \in D_\gamma^p\} + \sum\{w_x^q : x \in D_\gamma^q\} \\ &\leq 2 \cdot \max(\sum\{w_x^p : x \in D_\gamma^p\}, \sum\{w_x^q : x \in D_\gamma^q\}) \\ &\leq l_\gamma \end{aligned}$$

where the last inequality holds because  $p, q \in W$ .

- (c) For  $n$  such that  $l_\gamma \leq n < L$ , define  $s_x(n)$  as follows.

- (i)  $s_x(n) = \bigcup\{K_{z,n} : z \in D_{\leq x}^p \text{ or for some } z' \in D_\gamma^p \cap D_\gamma^q, z \in (D^p \cup D^q)_\gamma \text{ and } z \leq z' \leq x\}$ , if  $x \in D_\gamma^p \setminus D_\gamma^q$ ;
- (ii)  $s_x(n) = \bigcup\{K_{z,n} : z \in D_{\leq x}^q \text{ or for some } z' \in D_\gamma^p \cap D_\gamma^q, z \in (D^p \cup D^q)_\gamma \text{ and } z \leq z' \leq x\}$ , if  $x \in D_\gamma^q \setminus D_\gamma^p$ ;
- (iii)  $s_x(n) = \bigcup\{K_{z,n} : z \in (D^p \cup D^q)_{\leq x}\}$ , if  $x \in D_\gamma^p \cap D_\gamma^q$ .

Suppose  $E \subseteq D_\gamma^p$  is downward closed. Then

$$\begin{aligned} \bigcup\{s_x(n) : x \in E\} &= \bigcup\{K_{z,n} : z \in (D^p \cup D^q)_{\leq x} \text{ for some } x \in E \cap U\} \\ &\quad \cup \bigcup\{K_{z,n} : z \in E \text{ and for no } x \in E \cap U \text{ do we have } z \leq x\}. \end{aligned}$$

So

$$\begin{aligned} |\bigcup\{s_x(n) : x \in E\}| &\leq \sum\{|F_z^p| : z \in D_{\leq x}^p \text{ for some } x \in E \cap U\} \\ &\quad + \sum\{|F_z^q| : z \in D_{\leq x}^q \text{ for some } x \in E \cap U\} \\ &\quad + \sum\{|F_z^p| : z \in E \text{ and for no } x \in E \cap U \text{ do we have } z \leq x\} \\ &\leq 2k_{E \cap U} \\ &\quad + \sum\{|F_z^p| : z \in E \text{ and for no } x \in E \cap U \text{ do we have } z \leq x\} \\ &\leq \sum\{w_z^p : z \in D_{\leq x}^p \text{ for some } x \in E \cap U\} \\ &\quad + \sum\{w_z^p : z \in E \text{ and for no } x \in E \cap U \text{ do we have } z \leq x\} \\ &= \sum\{w_z^p : z \in E\}. \end{aligned}$$



Similarly, if  $E$  is a downward closed subset of  $D_\gamma^q$ , then  $|\bigcup\{s_x(n) : x \in E\}| \leq \sum\{w_x^q : x \in E\}$ .

(d) Let  $r_i = r^* \cup \{(s_x, w_x, F_x) : x \in D_\gamma^p \cup D_\gamma^q\}$ , where the triples  $(s_x, w_x, F_x)$  are obtained as follows.

- (i) Each  $s_x$  has domain  $L$ ,  $s_x \upharpoonright l_\gamma = s_x^p$  if  $x \in D_\gamma^p$  and  $s_x \upharpoonright l_\gamma = s_x^q$  if  $x \in D_\gamma^q$ . (This is unambiguous if both clauses hold because of item (4) in the list of properties of  $A'$ .) For  $l_\gamma \leq n < L$ ,  $s_x(n)$  is as defined in (c).
- (ii) We have  $w_x = w_x^p$  if  $x \in D_\gamma^p$  and  $w_x = w_x^q$  if  $x \in D_\gamma^q$  (and this is unambiguous if both clauses hold).
- (iii) For  $x \in D^p \setminus D^q$ ,  $F_x = F_x^p$ . For  $x \in D^q \setminus D^p$ ,  $F_x = F_x^q$ . For  $x \in D^p \cap D^q$ ,  $F_x = F_x^p \cup F_x^q$ .

We must check that  $r_i$  is as desired. First we check that  $r_i$  is a well-defined condition. In Definition 4.1, clause 1 and the first and third statements of clause 2 hold by definition. The second statement holds below rank  $\xi_i$  because  $r^*$  is a condition. At rank  $\gamma = \xi_i$ , it holds because for each  $x \in (D^p \cup D^q)_\gamma$  and  $n < L$ , if  $n < l_\gamma$  then  $|s_x(n)| \leq n$  because  $p$  and  $q$  are conditions and if  $l_\gamma \leq n < L$  then the argument at the end of (b) above shows that  $|s_x(n)| \leq l_\gamma \leq n$ . For the last statement, we have that  $|F_x|$  is bounded by one of  $|F_x^p|$ ,  $|F_x^q|$ ,  $|F_x^p| + |F_x^q|$ . In all cases, because  $p, q \in W$ , we have that  $|F_x|$  is bounded by either  $2 \cdot |F_x^p| \leq w_x^p = w_x$  or  $2 \cdot |F_x^q| \leq w_x^q = w_x$ . For clause 3, the property is inherited from  $r^*$  if the rank of  $x$  is less than  $\xi_i$ , and, if the rank of  $x$  is  $\xi_i$ , is inherited from  $p$  or  $q$  if  $\xi_i \in \bar{D}^p \setminus \bar{D}^q$  or  $\xi_i \in \bar{D}^q \setminus \bar{D}^p$ . Otherwise we have  $\sum\{w_x : x \in (D^p \cup D^q)_{\leq x}\} \leq \sum\{w_x^p : x \in D_{\leq x}^p\} + \sum\{w_x^q : x \in D_{\leq x}^q\} \leq l_{\xi_i} \leq L$ . Clause 4 is inherited from  $r^*$  at ranks below  $\xi_i$  and holds by definition at rank  $\xi_i$ .

Now we check that  $r$  extends  $p$  and  $q$ . By symmetry, it is enough to check that  $r$  extends  $p$ . All of the clauses 5–9 in the definition hold below rank  $\xi_i$  because  $r^* \leq r_{i-1} \leq p \upharpoonright \xi_{i-1} + 1$ . Consider now what they say at rank  $\gamma = \xi_i$ . The inclusion of the domains and all but the last part of 6 hold by definition of  $r$ . The last part of 6 holds because if  $x \in D_\gamma^p$ ,  $\dot{f} \in F_x^p$  and  $l_\gamma \leq n < L$ , we chose  $r^*$  so that  $r^* \Vdash_{\mathbb{N}_{<\gamma}} \dot{f}(n) \in K_{x,n} \subseteq s_x(n)$ . Because  $\dot{f}$  is a  $\mathbb{N}_x$ -name and  $\mathbb{N}_x$  is completely embedded in  $\mathbb{N}_{<\gamma}$ , it follows that  $r^* \upharpoonright x = r_i \upharpoonright x$  also forces  $\dot{f}(n) \in s_x(n)$ .

The proof of clause 7 is a case by case analysis. Suppose  $x, y \in D_\gamma^p$ ,  $x < y$  and  $l_\gamma \leq n < L$ . Each of  $x$  and  $y$  comes under either (c)(i) or

(c)(iii). Since the formulas used there are increasing functions of  $x$ , we need only consider the following two cases.

Case 1.  $x \in D_\gamma^p \setminus D_\gamma^q$  and  $y \in D_\gamma^p \cap D_\gamma^q$ . Let  $m \in s_x(n)$  and fix  $z$  witnessing this. (So, in particular,  $m \in K_{z,n}$ .) We will show that  $K_{z,n} \subseteq s_y(n)$ . If  $z \in D_{\leq x}^p$ , then also  $z \in D_{\leq y}^p$ , so  $K_{z,n} \subseteq s_y(n)$ . The other possibility is that for some  $z' \in D_\gamma^p \cap D_\gamma^q$ ,  $z \in (D^p \cup D^q)_\gamma$  and  $z \leq z' \leq x$ . Then  $z' \in (D^p \cup D^q)_{\leq y}$ , so again  $K_{z,n} \subseteq s_y(n)$ .

Case 2.  $x \in D_\gamma^p \cap D_\gamma^q$  and  $y \in D_\gamma^p \setminus D_\gamma^q$ . Fix  $z \in (D^p \cup D^q)_{\leq x}$ . Taking  $z' = x$ , we have  $z \leq z' < y$  witnessing that  $K_{z,n} \subseteq s_y(n)$ .

For clause 8, we have that  $\sum\{w_x : x \in (D^p \cup D^q)_\gamma\} \leq \sum\{w_\gamma^p : x \in D_\gamma^p\} + \sum\{w_\gamma^q : x \in D_\gamma^q\} = L - l_\gamma$  by the definition of  $L$  in (a). Finally, clause 9 was checked in (c).

For  $i = k - 1$ , we get that  $r_i$  is a common extension of  $p$  and  $q$ .

This complete the proof that  $N_Q$  is ccc.  $\square$

## 5 Proof of the main theorem

This section is devoted to the proof of Hechler's theorem for the null ideal. We will show that the forcing notion  $N_Q$  satisfies all the requirements of the theorem.

**Lemma 5.1.** *For a downward closed set  $B \subseteq Q$ ,  $p \in N_Q$ ,  $\xi \in \bar{D}^p$  and  $N < \omega$ , there is  $q \in N_B$  such that  $q \leq p$  and  $l_\xi^p \geq N$ .*

*Proof.* Just apply Lemma 4.7 to  $p' = p$  and  $N$ .  $\square$

**Lemma 5.2.** *For a downward closed set  $B \subseteq Q$ ,  $p \in N_B$  and  $a \in B$ , there is  $q \in N_B$  such that  $q \leq p$  and  $a \in D^q$ .*

*Proof.* We may assume that  $a \notin D^p$ . Let  $\alpha = \text{rank}(a)$ .

If  $\alpha \notin \bar{D}^p$ , then define  $q \in N_B$  by letting  $D^q = D^p \cup \{a\}$ ,  $s_a^q = \emptyset$ ,  $w_a^q = 0$ ,  $F_a^q = \emptyset$  and other components of  $q$  are the same as  $p$ .

Now we assume that  $\alpha \in \bar{D}^p$ . Define an  $\alpha$ -preextension  $p'$  of  $p$  in  $N_B$  by letting  $D^{p'} = D^p \cup \{a\}$ ,  $s_a^{p'}$  is arbitrary with length  $l_\alpha^p$ ,  $w_a^{p'} = 0$ ,  $F_a^{p'} = \emptyset$  and other components of  $p'$  are the same as  $p$ . Apply Lemma 4.7 to  $p$ ,  $p'$  and  $N = 0$ , and we get  $q \in N_B$  with  $q \leq p$  and  $a \in D^q$ .  $\square$

**Lemma 5.3.** *For a downward closed set  $B \subseteq Q$ ,  $p \in N_B$  and  $a \in D^p$ , there is  $q \in N_B$  such that  $q \leq p$  and  $w_a^q \geq |F_a^q| + 1$ .*

*Proof.* Let  $\alpha = \text{rank}(a)$ . Define an  $\alpha$ -preextension  $p'$  of  $p$  in  $N_B$  by letting  $D^{p'} = D^p$ ,  $w_a^{p'} = w_a^p + 1$  and other components of  $p'$  are the same as  $p$ . Apply Lemma 4.7 to  $p$ ,  $p'$  and  $N = 0$ , and we get  $q \in N_B$  as required.  $\square$

**Lemma 5.4.** *For a downward closed set  $B \subseteq Q$ ,  $p \in N_B$ ,  $a \in D^p$  and an  $N_a$ -name  $\dot{f}$  for a function in  $\omega^\omega$ , there is  $q \in N_B$  such that  $q \leq p$  and  $\dot{f} \in F_a^q$ .*

*Proof.* First use Lemma 5.3, and then put  $\dot{f}$  into  $F_a^q$ .  $\square$

Let  $\mathbf{V}$  be a ground model and  $G$  an  $N_Q$ -generic filter over  $\mathbf{V}$ . For  $a \in Q$ , let  $G \restriction a = G \cap N_a = \{p \restriction a : p \in G\}$ . Then  $G \restriction a$  is an  $N_a$ -generic filter over  $\mathbf{V}$ .

In  $\mathbf{V}[G]$ , for  $a \in Q$  let  $\varphi_a = \bigcup \{s_a^p : p \in G \text{ and } a \in D^p\}$ . By Lemmata 5.1 and 5.2,  $\varphi_a$  is defined for every  $a \in Q$ , and belongs to  $\mathcal{S}$ .

**Lemma 5.5.** *In  $\mathbf{V}[G]$ , for every  $a \in Q$  and  $f \in \omega^\omega \cap \mathbf{V}[G \restriction a]$ , for all but finitely many  $n < \omega$  we have  $f(n) \in \varphi_a(n)$ .*

*Proof.* Follows from Lemma 5.4 and the definition of  $N_Q$ .  $\square$

**Lemma 5.6.** *For  $a, b \in Q$ , if  $a < b$  and  $\text{rank}(a) = \text{rank}(b)$ , then for all but finitely many  $n < \omega$  we have  $\varphi_a(n) \subseteq \varphi_b(n)$ .*

*Proof.* Clear from the definition of  $N_Q$ .  $\square$

For  $a \in Q$ , let  $H_a = H_{\varphi_a}$ . Then each  $H_a$  is a null subset of  $2^\omega$ . We will show that, in  $\mathbf{V}[G]$ , the set  $\{H_a : a \in Q\}$  is order-isomorphic to  $(Q, \leq)$  and cofinal in  $(\mathcal{N}, \subseteq)$ .

**Lemma 5.7.** *Let  $a \in Q$ . For a Borel null set  $X \subseteq 2^\omega$  which is coded in  $\mathbf{V}[G \restriction a]$ , we have  $X \subseteq H_a$ .*

*Proof.* Follows from Lemma 5.5 and the observation in Section 2.  $\square$

**Lemma 5.8.** *In  $\mathbf{V}[G]$ , for every null set  $X \subseteq 2^\omega$  there is  $a \in Q$  satisfying  $X \subseteq H_a$ .*

*Proof.* We may assume that  $X$  is a Borel set in  $\mathbf{V}[G]$ . By our assumption on  $(Q, \leq)$  and because  $N_Q$  satisfies ccc,  $X$  is coded in  $\mathbf{V}[G \restriction a]$  for some  $a \in Q$ , and by Lemma 5.7, we have  $X \subseteq H_a$ .  $\square$

**Lemma 5.9.** *For  $a, b \in Q$ , if  $a \leq b$  then  $H_a \subseteq H_b$ .*

*Proof.* If  $a \ll b$ , then  $H_a$  is coded in  $\mathbf{V}[G \restriction b]$  and hence  $H_a \subseteq H_b$  follows from Lemma 5.7. If  $a < b$  and  $\text{rank}(a) = \text{rank}(b)$ , then it follows from Lemma 5.6 and the observation in Section 2.  $\square$

For each  $a \in Q$ , let  $r_a = r_{\varphi_a}$  and  $R_a = R_{\varphi_a}$  as defined in Section 2. As we observed in Section 3, we define an  $\mathbb{N}_Q$ -name  $\dot{r}_a$  for  $r_a$  so that, for  $p \in \mathbb{N}_Q$  if  $a \in D^p$  and  $|s_a^p| = n$  then  $p$  decides the value of  $\dot{r}_a \upharpoonright n$ .

**Lemma 5.10.** *For  $a, b \in Q$ , if  $a \not\leq b$  then  $H_a \not\subseteq H_b$ .*

*Proof.* Suppose that  $a \not\leq b$ . Since we always have  $R_b \cap H_b = \emptyset$  and  $R_b \neq \emptyset$ , it suffices to show that  $R_b \subseteq H_a$ .

Fix  $p \in \mathbb{N}_Q$  and  $M < \omega$ . By Lemmata 5.2 and 5.3, we may assume that  $a, b \in D^p$  and  $w_a^p \geq |F_a^p| + 1$ .

We will find  $q \leq p$  and  $m > M$  which satisfy  $q \Vdash \dot{r}_b(m) \in s_a^q(m)$ . This implies that for infinitely many  $m < \omega$  we have  $r_b(m) \in \varphi_a(m)$ , and hence  $R_b \subseteq H_a$ .

Let  $\alpha = \text{rank}(a)$ ,  $\beta = \text{rank}(b)$ ,  $B = \{x \in Q : x \leq b\}$ . Note that  $a \notin B$  by the assumption. Extend  $p$  if necessary to arrange the following.

If  $B_\alpha \neq \emptyset$ , then  $B_\alpha \cap D^p \neq \emptyset$ .

(The following observation is not used in the proof, but note for clarity that because of the definition of rank for elements of  $Q \setminus R$ , the ranks of the elements of a downward closed set need not be an initial segment of the ordinals. For example, if  $R = \omega_1$  ordered as usual and  $Q$  is  $R$  with new elements  $e_\alpha$ , where  $e_\alpha \leq \alpha$  but no other relations hold other than the ones needed to ensure transitivity, then  $e_\alpha$  has rank  $\alpha$  and every subset of  $\{e_\alpha : \alpha < \omega_1\}$  is downward closed. Thus the assumption  $B_\alpha \neq \emptyset$  can fail even if  $\alpha < \beta$ .)

We set  $m = \max\{M, l_\alpha^p\} + 1$ .

Using Lemma 5.1, get  $p^* \in \mathbb{N}_B$  extending  $p \upharpoonright B$  such that  $|s_b^{p^*}| \geq m + 1$ . By the choice of  $\dot{r}_b$ ,  $p^*$  decides the value of  $\dot{r}_b(m)$ , so let  $k$  be such that  $p^* \Vdash_{\mathbb{N}_B} \dot{r}_b(m) = k$ .

We will construct  $q \in \mathbb{N}_Q$  satisfying  $q \leq p$  and  $q \leq p^*$ , using an argument similar to, but somewhat more difficult than, the proof of Lemma 4.5.

The proof which follows is really two similar but different proofs, one for the case where  $B_\alpha \neq \emptyset$  and one for the case  $B_\alpha = \emptyset$ . In order to be able to write as much as possible of the two proofs as one, we will use the abuse of notation  $\max\{l_\alpha^{p^*}, l_\alpha^p\}$  to designate  $l_\alpha^{p^*}$  when  $B_\alpha \neq \emptyset$  and  $l_\alpha^p$  when  $B_\alpha = \emptyset$  (in which case  $l_\alpha^{p^*}$  is actually not defined).

We will be done if we build  $q \leq p$  with  $k \in s_a^q(m)$ . For  $x \in D_\alpha^{p^*}$ , let  $(s_x, w_x, F_x) = (s_x^{p^*}, w_x^{p^*}, F_x^{p^*})$ . For  $x \in D_\alpha^p \setminus D_\alpha^{p^*}$ , let  $(s_x, w_x, F_x) = (s_x^p, w_x^p, F_x^p)$ . Let

$$L = \sum\{w_x : x \in D_\alpha^p \cup D_\alpha^{p^*}\} + \max\{l_\alpha^{p^*}, l_\alpha^p\} + m + 1.$$

Choose  $q_0 \in N_\alpha$  so that  $q_0 \leq p \restriction \alpha$ ,  $q_0 \leq p^* \restriction \alpha$  (and hence also  $q_0 \restriction B_{<\alpha} \leq p^* \restriction \alpha$ ), and  $q_0$  decides the values of  $\dot{f} \restriction L$  for all  $\dot{f} \in \bigcup \{F_x : x \in D_\alpha^p \cup D_\alpha^{p^*}\}$ . For  $x \in D_\alpha^p \cup D_\alpha^{p^*}$  and  $n \in L \setminus |s_x|$ , let  $K_{x,n} \subseteq \omega$  be the set satisfying  $q_0 \Vdash K_{x,n} = \{\dot{f}(n) : \dot{f} \in F_x\}$ . For  $x \in D_\alpha^p \cup D_\alpha^{p^*}$  and  $n \in L \setminus |s_x|$ , if  $(x, n) \neq (a, m)$  then let  $K'_{x,n} = K_{x,n}$ , and let  $K'_{a,m} = K_{a,m} \cup \{k\}$ . By the assumption that  $w_\alpha^p \geq |F_\alpha^p| + 1$ , we have  $|K'_{x,n}| \leq w_x$  for all  $x \in D_\alpha^p \cup D_\alpha^{p^*}$  and  $n \in L \setminus |s_x|$ .

Define  $s_x^*$  for  $x \in D_\alpha^p \cup D_\alpha^{p^*}$  as follows. If  $x \in D_\alpha^{p^*}$ , then  $|s_x^*| = L$ ,  $s_x^* \restriction l_\alpha^{p^*} = s_x$ , and for  $n \in L \setminus l_\alpha^{p^*}$ ,

$$s_x^*(n) = \bigcup \{K'_{z,n} : z \in D_{\leq x}^{p^*}\}.$$

If  $x \in D_\alpha^p \setminus D_\alpha^{p^*}$ , then  $|s_x^*| = L$ ,  $s_x^* \restriction l_\alpha^p = s_x$ , and for  $n \in L \setminus l_\alpha^p$ ,

$$s_x^*(n) = \begin{cases} \bigcup \{s_z(n) : z \in D_{\leq x}^p \cap D_\alpha^{p^*}\} \cup \bigcup \{K'_{z,n} : z \in D_{\leq x}^p \setminus D_\alpha^{p^*}\}, & l_\alpha^p \leq n < \max\{l_\alpha^{p^*}, l_\alpha^p\} \\ \bigcup \{K'_{z,n} : z \in (D_\alpha^p \cup D_\alpha^{p^*})_{\leq x}\}, & \max\{l_\alpha^{p^*}, l_\alpha^p\} \leq n < L \end{cases}$$

Define  $q_1$  by  $q_1 = \{(s_x^{q_1}, w_x^{q_1}, F_x^{q_1}) : x \in D^{q_0} \cup D^{p^*} \cup D_\alpha^p\}$  where

1. For  $x \in D^{q_0}$ ,  $(s_x^{q_1}, w_x^{q_1}, F_x^{q_1}) = (s_x^{q_0}, w_x^{q_0}, F_x^{q_0})$
2. For  $x \in D_\alpha^p \cup D_\alpha^{p^*}$ ,  $(s_x^{q_1}, w_x^{q_1}, F_x^{q_1}) = (s_x^*, w_x, F_x)$
3. For  $x \in D^{p^*} \setminus Q_{<\alpha+1}$ ,  $(s_x^{q_1}, w_x^{q_1}, F_x^{q_1}) = (s_x^{p^*}, w_x^{p^*}, F_x^{p^*})$

We now check that  $q_1 \in N_Q$ . The conditions of Definition 4.1 are satisfied below (resp. above) rank  $\alpha$  because  $q_0$  (resp.  $p^*$ ) is a condition. Consider what they say at rank  $\alpha$ . The first clause is trivial. The fourth holds because the  $s_x^{q_1}$ 's all have domain  $L$ . The third clause can be checked in two cases.

- (i) If  $x \in D_\alpha^{p^*}$ , then  $D_{\leq x}^{q_1} = (D^p \cup D^{p^*})_{\leq x} = D_{\leq x}^{p^*}$ , so  $\sum \{w_z^{q_1} : z \in D_{\leq x}^{q_1}\} = \sum \{w_z^{p^*} : z \in D_{\leq x}^{p^*}\} \leq l_\alpha^{p^*} \leq L$ .
- (ii) If  $x \in D_\alpha^p \setminus D_\alpha^{p^*}$ , then  $D_{\leq x}^{q_1} = D_{\leq x}^p \cup D_{\leq x}^{p^*}$ , so  $\sum \{w_z^{q_1} : z \in D_{\leq x}^{q_1}\} = \sum \{w_z : z \in D_{\leq x}^p \cup D_{\leq x}^{p^*}\} \leq \sum \{w_z : z \in D_\alpha^p \cup D_\alpha^{p^*}\} \leq L$ .

For the second, all the requirements except that the  $s_x^{q_1}$ 's are partial slaloms follow from the fact that  $p$  and  $p^*$  are conditions. We need to check that  $|s_x^*(n)| \leq n$  for each relevant  $n$ . If  $x \in D_\alpha^{p^*}$ , then for  $l_\alpha^{p^*} \leq n < L$ , we have  $|s_x^*(n)| \leq \sum \{w_z^{p^*} : z \in D_{\leq x}^{p^*}\} \leq |s_x^{p^*}| = l_\alpha^{p^*} \leq n$ . If  $x \in D_\alpha^p \setminus D_\alpha^{p^*}$ , we consider three cases.

Case 1.  $l_\alpha^p \leq n < \max\{l_\alpha^{p^*}, l_\alpha^p\}$ . In order for this case to be non vacuous, we must have  $\alpha \in \bar{D}^{pB}$ . Then Definition 4.1(9) for  $p^* \leq p \restriction B$  with  $E = D_{\leq x}^p \cap D^{p^*}$  gives

$$\begin{aligned} |s_x^*(n)| &\leq \sum\{w_z^p : z \in E\} + (n - l_\alpha^p) + \sum\{w_z^p : z \in D_{\leq x}^p \setminus E\} \\ &= \sum\{w_z^p : z \in D_{\leq x}^p\} + (n - l_\alpha^p) \\ &\leq l_\alpha^p + (n - l_\alpha^p) = n. \end{aligned}$$

Case 2.  $\max\{l_\alpha^{p^*}, l_\alpha^p\} \leq n < L$ . If  $\alpha \in \bar{D}^{pB}$ , then Definition 4.1(8) for  $p^* \leq p \restriction B$  gives

$$\sum\{w_z^{p^*} : z \in D_\alpha^{p^*}\} \leq \sum\{w_z^p : z \in D_\alpha^{pB}\} + (l_\alpha^{p^*} - l_\alpha^p).$$

Removing terms with  $z \not\leq x$  from both sides (see Remark 1) gives

$$\sum\{w_z^{p^*} : z \in D_{\leq x}^{p^*}\} \leq \sum\{w_z^p : z \in D_{\leq x}^p \cap B\} + (l_\alpha^{p^*} - l_\alpha^p).$$

From the formula for  $s_x(n)$  we now get

$$\begin{aligned} |s_x(n)| &\leq \sum\{w_z^{p^*} : z \in D_{\leq x}^{p^*}\} + \sum\{w_z^p : z \in D_{\leq x}^p \setminus B\} \\ &\leq \sum\{w_z^p : z \in D_{\leq x}^p \cap B\} + (l_\alpha^{p^*} - l_\alpha^p) + \sum\{w_z^p : z \in D_{\leq x}^p \setminus B\} \\ &= \sum\{w_z^p : z \in D_{\leq x}^p\} + (l_\alpha^{p^*} - l_\alpha^p) \\ &\leq l_\alpha^p + (l_\alpha^{p^*} - l_\alpha^p) = l_\alpha^{p^*} \leq n. \end{aligned}$$

If  $\alpha \notin \bar{D}^{pB}$ , then  $B_\alpha = \emptyset$ , so  $\alpha \notin \bar{D}^{p^*}$ . The formula for  $s_x^*(n)$  thus reduces to  $s_x^*(n) = \bigcup\{K'_{z,n} : z \in D_{\leq x}^p\}$ , and hence  $|s_x^*(n)| \leq \sum\{w_z^p : z \in D_{\leq x}^p\} \leq l_\alpha^p \leq n$ .

Thus,  $q_1$  is a condition. We now check Definition 4.1(5–9) for  $q_1 \leq p^*$  and  $q_1 \leq p \restriction B \cup Q_{<\alpha+1}$ . (We only need the latter, but the former is needed at one point of the proof.) Clause 5 follows from the definition of  $q_1$ . For clauses 6–9, first note that below rank  $\alpha$ , they hold because  $q_0 \leq p \restriction \alpha$  and  $q_0 \leq p^* \restriction \alpha$ . Consider what happens at rank  $\alpha$ . Clause 6 holds because for  $x \in D_\alpha^p \cup D_\alpha^{p^*}$  and all the relevant values of  $\dot{f}$  and  $n$ , we have from the definitions that  $q_0 \Vdash \dot{f}(n) \in K_{x,n}$  and  $K_{x,n} \subseteq s_x^*(n)$ . For clause 7, we consider three cases. Let  $x < y$  be elements of  $D_\alpha^p \cup D_\alpha^{p^*}$ .

- (i) If  $x, y \in D_\alpha^{p^*}$ , then for checking  $q_1 \leq p^*$ , just use the monotonicity of  $s_x^*(n)$  as a function of  $x$ . For checking  $q_1 \leq p \restriction B \cup Q_{<\alpha+1}$ , we also need to consider values of  $n$  such that  $l_\alpha^p \leq n < l_\alpha^{p^*}$ . But then  $s_x^*(n) = s_x^{p^*}(n) \subseteq s_y^{p^*}(n) = s_y^*(n)$  because  $p^* \leq p \restriction B$ .

This is the only case to consider for checking clause 7 for  $q_1 \leq p^*$  at stage  $\alpha$ . The remaining cases deal with checking  $q_1 \leq p \restriction B \cup Q_{<\alpha+1}$ . Note that if  $y \in D_\alpha^{p^*} \cap D_\alpha^p = D_\alpha^p \cap B$  then also  $x \in D_\alpha^{p^*} \cap D_\alpha^p$  since  $B$  is downward closed.

- (ii) If  $x, y \in D_\alpha^p \setminus D_\alpha^{p^*}$ , use the monotonicity of  $s_x^*(n)$  as a function of  $x$ .
- (iii) If  $x \in D_\alpha^{p^*} \cap D_\alpha^p$  and  $y \in D_\alpha^p \setminus D_\alpha^{p^*}$ , then consider first a value of  $n$  such that  $l_\alpha^p \leq n < l_\alpha^{p^*}$ . We have  $s_x^*(n) = s_x(n) \subseteq \bigcup \{s_z(n) : z \in D_{\leq y}^p \cap D_\alpha^{p^*}\} \subseteq s_y^*(n)$ . Next consider  $n$  such that  $l_\alpha^{p^*} \leq n < L$ . We have  $s_x^*(n) = \bigcup \{K'_{z,n} : z \in D_{\leq x}^{p^*}\} \subseteq \bigcup \{K'_{z,n} : z \in (D_\alpha^p \cup D_\alpha^{p^*})_{\leq y}\} = s_y^*(n)$ .

That takes care of clause 7. Clause 8 follows from the fact that if  $\alpha \in \bar{D}^{p^*}$ , then from the definition of  $L$  we have  $\sum \{w_x : x \in D_\alpha^{p^*} \cup D_\alpha^p\} \leq L - l_\alpha^{p^*}$ , and if  $\alpha \in \bar{D}_\alpha^p \setminus \bar{D}_\alpha^{p^*}$ , then  $\sum \{w_x : x \in D_\alpha^{p^*} \cup D_\alpha^p\} \leq L - l_\alpha^p$ . For clause 9, first we check  $q_1 \leq p^*$ . If  $\alpha \in \bar{D}^{p^*}$ ,  $E \subseteq D_\alpha^{p^*}$  is downward closed in  $D_\alpha^{p^*}$  and  $l_\alpha^{p^*} \leq n < L$ , then  $|\bigcup \{s_x^*(n) : x \in E\}| = |\bigcup \{K'_{x,n} : x \in E\}| \leq \sum \{w_x^{p^*} : x \in E\}$ . Next we check  $q_1 \leq p \upharpoonright B \cup Q_{<\alpha+1}$ . Note that the elements of rank  $\alpha$  are the same for the domains of  $p$  and  $p \upharpoonright B \cup Q_{<\alpha+1}$ . Also  $\alpha \in D_\alpha^p$  since  $a \in D^p$ . Let  $E \subseteq D_\alpha^p$  be downward closed. Consider two cases.

Case 1.  $l_\alpha^p \leq n < l_\alpha^{p^*}$ . We have

$$\begin{aligned} |\bigcup \{s_x^*(n) : x \in E\}| &= |\bigcup \{s_x^*(n) : x \in E \cap B\} \cup \bigcup \{s_x^*(n) : x \in E \setminus B\}| \\ &= |\bigcup \{s_x^{p^*}(n) : x \in E \cap B\} \cup \bigcup \{K'_{x,n} : x \in E \setminus B\}| \\ &\leq \sum \{w_x^{p^*} : x \in E \cap B\} + (n - l_\alpha^p) + \sum \{w_x^p : x \in E \setminus B\} \\ &= \sum \{w_x^p : x \in E\} + (n - l_\alpha^p). \end{aligned}$$

Case 2.  $\max\{l_\alpha^{p^*}, l_\alpha^p\} \leq n < L$ . Let  $E' = \{z \in D_\alpha^{p^*} : \text{for some } x \in E, z \leq x\}$ . We have

$$\begin{aligned} |\bigcup \{s_x^*(n) : x \in E\}| &= |\bigcup \{s_x^*(n) : x \in E \cap B\} \cup \bigcup \{s_x^*(n) : x \in E \setminus B\}| \\ &= |\bigcup \{K'_{x,n} : x \in E'\} \cup \bigcup \{K'_{x,n} : x \in E \setminus B\}| \\ &\leq \sum \{w_x^{p^*} : x \in E'\} + \sum \{w_x^p : x \in E \setminus B\}. \end{aligned}$$

If  $E'$  is empty, then this last expression is  $\leq \sum \{w_x^p : x \in E\}$ . If not, then Definition 4.1(8) applied to  $p^* \leq p \upharpoonright B$  (with terms outside  $E'$  eliminated from both sides) gives that

$$\begin{aligned} &\sum \{w_x^{p^*} : x \in E'\} + \sum \{w_x^p : x \in E \setminus B\} \\ &\leq \sum \{w_x^p : x \in E \cap B\} + (l_\alpha^{p^*} - l_\alpha^p) + \sum \{w_x^p : x \in E \setminus B\} \\ &= \sum \{w_x^p : x \in E\} + (l_\alpha^{p^*} - l_\alpha^p) \\ &\leq \sum \{w_x^p : x \in E\} + (n - l_\alpha^p). \end{aligned}$$

Thus, the conditions for  $q_1 \leq p^*$  and  $q_1 \leq p \restriction B \cup Q_{<\alpha+1}$  hold up to rank  $\alpha$ . Above rank  $\alpha$ ,  $q_1$  agrees with  $p^*$ , so Definition 4.1(6–9) hold trivially for  $q_1 \leq p^*$ . For  $q_1 \leq p \restriction B \cup Q_{<\alpha+1}$  we need to prove the clauses for  $\xi > \alpha$ . All of them follow from the fact that  $p^* \leq p \restriction B$ ,  $q_1 \restriction \xi \leq p^* \restriction \xi$ , and  $q_1$  agrees with  $p^*$  at rank  $\xi$ . (The fact that  $q_1 \restriction \xi \leq p^* \restriction \xi$  is used to check the last part of clause 6.)

Now we apply Lemma 4.5 to  $p$  and  $q_1$ , and we get  $q \in \mathbb{N}_Q$  such that  $q \leq p$  and  $q \Vdash \dot{r}_b(m) \in s_a^q(m)$ .  $\square$

Now we have the following main theorem.

**Theorem 5.11.** *Let  $\mathcal{N}$  be the collection of null sets in  $2^\omega$ . Suppose that  $Q$  is a partially ordered set such that every countable subset of  $Q$  has a strict upper bound in  $Q$ . Then in the forcing model by  $\mathbb{N}_Q$ ,  $(\mathcal{N}, \subseteq)$  contains a cofinal subset  $\{H_a : a \in Q\}$  which is order-isomorphic to  $(Q, \leq)$ , that is,*

1. *for every  $X \in \mathcal{N}$  there is  $a \in Q$  such that  $X \subseteq H_a$ , and*
2. *for  $a, b \in Q$ ,  $H_a \subseteq H_b$  if and only if  $a \leq b$ .*

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